THE RELATION BETWEEN LEARNERS' SPONTANEOUS FOCUSING ON QUANTITATIVE RELATIONS AND THEIR RATIONAL NUMBER KNOWLEDGE

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Abstract: Many difficulties learners have with rational number tasks can be attributed to the "natural number bias", i.e. the tendency to inappropriately use natural number properties in rational numbers tasks (Van Hoof, 2015). McMullen and colleagues found a relevant source of individual differences in the learning of those aspects of rational numbers that are susceptible to the natural number bias, namely Spontaneous Focusing On quantitative Relations (SFOR) (McMullen, 2014). While McMullen and colleagues showed that SFOR relates to rational number knowledge as a whole, we studied its relation with several aspects of the natural number bias. Additionally, we 1) included test items addressing operations with rational numbers and 2) controlled for general mathematics achievement and age. Results showed that SFOR related strongly to rational number knowledge, even after taking into account several control variables. Results are discussed for each of the three aspects of the natural number bias separately.

Key words: spontaneous focusing on quantitative relations, fraction, rational number, elementary school, natural number bias

Introduction

Rational Number Knowledge

There is a broad agreement in the literature that a good understanding of rational numbers is of critical importance for mathematics achievement in general and for performance in specific domains of the mathematics curriculum in particular (Siegler et al., 2012). For example, Siegler, Thompson, and Schneider (2011) found high correlations (all between .54 and .86) between three measures of fraction magnitude knowledge (0-1 fraction number line estimation, 0-5 fraction number line estimation, and 0-1 fraction magnitude comparison) and general mathematics achievement in upper elementary school learners. This finding was replicated by Torbeyns, Schneider, Xin, and Siegler (2015) in three countries from different continents. Similar findings emerged from a recent study of Siegler et al. (2012), who concluded that fifth graders' rational number understanding predicted their overall mathematics and algebra scores in high school, even after controlling for reading achievement, IQ, working memory, whole number knowledge, family income, and family education.

Note. Jo Van Hoof and Tine Degrande contributed equally to this work and are both first authors.

Despite the critical importance of a good rational number knowledge, a large body of literature reported that children and even adults have a lot of difficulties dealing with various aspects of rational numbers (Bailey, Siegler, & Geary, 2014; Cramer, Post, & delMas, 2002; Li, Chen, & An, 2009; Mazzocco & Devlin, 2008; Merenluoto & Lehtinen, 2004; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vamvakoussi & Vosniadou, 2010; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). To give one example, more than one third of a representative sample of Flemish sixth graders did not reach the educational standards for rational numbers (Janssen, Verschaffel, Tuerlinckx, Van den Noortgate, & De Fraine, 2010).

The difficulties learners have with rational number tasks are often – at least in part – attributed to the "natural number bias" (Vamvakoussi et al., 2012; see Ni & Zhou, 2005, for the closely related idea of "whole number bias"), which is the tendency to inappropriately use natural number properties in rational numbers tasks (Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). Before learners are introduced to rational numbers in the classroom, they have already formed an idea of what a number is. This idea is based on their experiences (both in daily life and in school) with natural numbers. Once the learners are then instructed about rational numbers, the properties of natural numbers are not always applicable anymore, leading to problems and misconceptions with rational numbers (Vamvakoussi & Vosniadou, 2010). This becomes apparent in learners' systematic mistakes, specifically in rational number tasks where reasoning purely in terms of natural numbers results in an incorrect solution – these tasks are called incongruent. At the same time, much higher accuracy levels are found in rational number tasks where reasoning in terms of natural numbers leads to a correct answer – these tasks are called congruent. The vast literature on this natural number bias reports three main aspects that elicit such systematic errors. The first aspect relates to the density of the set of rational numbers. While natural numbers are characterized by a discrete structure (one can always indicate which number follows a given number; for example after 13 comes 14), rational numbers are characterized by a dense structure (you cannot say which number comes next, because between any two given rational numbers are always infinitely many other rational numbers) (e.g., Merenluoto & Lehtinen, 2004). The second aspect relates to the size of rational numbers. Research indicates that errors in size comparison tasks are repeatedly made because students incorrectly assume that, as is the case with natural numbers, "longer decimals are larger, shorter decimals are smaller", or "that a fraction's numerical value always increases when its denominator, numerator, or both increase" (Mamede, Nunes, & Bryant, 2005; Meert, Grégoire, & Noël, 2010; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Resnick et al., 1989). The third aspect concerns the effects of arithmetic operations on rational numbers. Several properties related to operations with natural numbers no longer consistently apply to rational numbers. Numerous studies report, for example, the well-known misconception of learners that "multiplication and addition will always lead to a larger outcome" and "division and subtraction will always result in a smaller outcome" (Hasemann, 1981; Vamvakoussi

et al., 2012; Van Hoof, Vandewalle et al., 2015).

While recent studies have begun to shed light on the causes of individual differences in rational number knowledge as a whole (Bailey et al., 2014), little is known about the causes of individual differences with regard to overcoming the natural number bias (Van Hoof, 2015; Van Hoof, Verschaffel, & Van Dooren, 2015). In this respect, McMullen and colleagues recently found a potentially relevant source of individual differences in the learning of rational numbers, namely learners' Spontaneous Focusing On quantitative Relations (SFOR) (e.g., McMullen, 2014; McMullen, Hannula-Sormunen, & Lehtinen, 2014; McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2015).

Spontaneous Focusing On Quantitative Relations (SFOR)

SFOR is described as "the spontaneous (i.e., undirected) focusing of attention on quantitative relations and the use of these relations in reasoning" (McMullen, Hannula-Sormunen, & Lehtinen, 2011, p. 218). SFOR belongs to the larger category of the "spontaneous quantitative focusing tendencies", as originally investigated by Hannula-Sormunen, Lehtinen and colleagues (e.g., Hannula & Lehtinen, 2005; Hannula, Lepola, & Lehtinen, 2010) and recently also by others (Edens & Potter, 2013; Kucian et al., 2012; Batchelor, Gilmore, & Inglis, 2015). A central idea underlying research on these tendencies is that there are not only individual differences in how learners reason about mathematics and use their numerical skills in formal mathematics learning situations, but also how often they spontaneously focus on mathematical aspects of informal everyday situations that are non-explicitly mathematical in nature. In these informal situations, the recognition and use of quantitative aspects in the situation is done at one's own initiative, so undirected and spontaneous (e.g., Hannula & Lehtinen, 2005; McMullen et al., 2011, 2014; McMullen, Hannula-Sormunen, & Lehtinen, 2013). Studies on those spontaneous quantitative focusing tendencies thus examine how often learners spontaneously use their number recognition and quantitative reasoning skills in situations where they are not explicitly guided to do so (Hannula & Lehtinen, 2005; McMullen, 2014).

The first spontaneous quantitative focusing tendency introduced by Hannula and Lehtinen (2001; 2005) in research with young learners (3.5-6 year olds) is Spontaneous Focusing On Numerosity (SFON). SFON is defined as the unguided, i.e. self-initiated recognition and use of exact numerosity in non-explicitly mathematical situations (Hannula & Lehtinen, 2005), and has been frequently studied. Those studies revealed that learners differ with respect to SFON (Batchelor et al., 2015; Edens & Potter, 2013; Hannula & Lehtinen, 2005; Hannula, Räsänen, & Lehtinen, 2007; Hannula et al., 2010), and that SFON is positively, and domain-specifically related to the development of enumeration, subitizing, number sequence, and basic arithmetic skills (Edens & Potter, 2013; Hannula & Lehtinen, 2005; Hannula et al., 2007, 2010), as well as to substantially later general mathematical skills and rational number understanding (Hannula-Sormunen, Lehtinen, & Räsänen, 2015; McMullen, Hannula-Sormunen, & Lehtinen, 2015).

Students with low mathematical skills have lower SFON tendency than students with average mathematical skills (Kucian et al.,

2012). Importantly, it has been demonstrated that children who do not focus on numerosity in the SFON tasks are able to recognize and produce quantities in the tasks once their attention is deliberately guided towards the exact number of items in the task. This indicates that SFON is a spontaneous attentional process, which is not entirely explained by children's numerical or other cognitive abilities needed for the task (Hannula & Lehtinen, 2005).

The research team of Lehtinen and Hannula-Sormunen recently expanded their SFON-studies and investigated whether there also exists a mathematically more advanced spontaneous quantitative focusing tendency in older learners (5-10 years of age), namely SFOR. In the context of these investigations, SFOR is a more mathematically complex spontaneous quantitative focusing tendency, in which learners' spontaneous focus lies on quantitative relations between two or more (sub-)sets, rather than merely on the numerosity of a single set. This is the core difference between SFOR and SFON. For example, a student may notice that she has walked already one third of her way to school, or has eaten three quarters of her chocolate bar, while another student may notice that he has walked one kilometer, or that he has eaten only three pieces of the chocolate bar. McMullen and colleagues revealed individual differences in learners' SFOR tendency (e.g., McMullen, 2014; McMullen et al., 2014, 2015). Those differences could not entirely be explained by learners' ability to recognize quantitative relations. This finding indicates that, in addition to learners' skills to recognize quantitative relations, differences in SFOR tendency could explain task performance (McMullen et al., 2014).

In recent years, McMullen and colleagues investigated in several studies the relation between learners' SFOR and their rational number knowledge and found repeatedly that learners' SFOR correlates highly with their rational number knowledge (see for more details McMullen, 2014; and McMullen et al., 2015). More specifically, they suggested that "the natural number bias may be related to spontaneous quantitative focusing tendencies" (McMullen, 2014, p. 46). This would implicate that SFOR correlates highly with performance, especially on incongruent rational number tasks (see definition above). However, McMullen and colleagues only looked at rational number knowledge as a whole – as measured by learners' total score on incongruent rational number knowledge – without focusing on those specific aforementioned aspects of the natural number bias separately.

The Present Study

We extended the above findings of McMullen and colleagues by further exploring the relationship between learners' SFOR and their rational number knowledge. More specifically, while McMullen and colleagues merely showed this relation for rational number knowledge as a whole, we tested whether spontaneous quantitative focusing tendencies may be related to all aspects of the natural number bias separately – besides rational number knowledge as a whole. Next to this general aim, we expanded previous SFOR-research in two ways. First, the test instruments used in the studies of McMullen and colleagues (e.g., McMullen et al., 2014, 2015) contained only items about the aspect of density and size, and not about the third aforementioned aspect of the natural num-

ber bias, namely operations. We aimed to study whether the relation between SFOR and rational number knowledge would not only hold for the density and size aspects, but also for operations with rational numbers. Second, next to the control variables that McMullen and colleagues (e.g., McMullen, 2014) already used, i.e. arithmetical fluency and non-verbal intelligence, we additionally took into account general mathematics achievement and age. With respect to the mathematics achievement variable, up to now, no study determined whether the impact of SFOR on rational number knowledge was due to its relation to a more general mathematical ability, or whether SFOR was a unique and direct predictor of this knowledge, in addition to general mathematical ability. In other words, it might have been possible that the relation between SFOR and rational number knowledge was mediated by general mathematics achievement. Regarding the age variable, both SFOR and rational number knowledge have been found to increase with age (see above). Thus, the observed relation between these two variables may be explained by schooling and/or maturation. Therefore, we added learners' age as a predictor in the model, when predicting rational number knowledge by means of SFOR.

Method

Participants

Participants were 356 Flemish learners from fourth to sixth grade (150 fourth graders, 97 fifth graders, 109 sixth graders) from five different primary schools. Approximately the same number of boys and girls participated in the study.

Tasks

SFOR measure: The Teleportation Task

We used the Teleportation Task of McMullen and colleagues (McMullen, 2014; McMullen et al., 2015) in order to measure SFOR (see Figure 1a). The Teleportation Task involved a cover story, which told that a set of supplies containing three sets of objects was sent from earth through space with a teleportation machine. However, when doing so, the objects got transformed in a number of ways (e.g., size, shape, color, number of items). Learners were first asked to describe the transformation in their own words in as many ways as possible. Second, they were shown a different numerosity of the same objects and they were asked to draw what they would expect to arrive based on the previous time. All learners were presented with four trials in total: two trials in which they were asked to describe the transformation (open-ended tasks), and two trials in which they were asked to draw the transformation (drawing tasks). When describing or drawing the transformation, learners could pay attention to the various non-mathematical changes (e.g., in terms of the colors, shapes of the objects), but also to the quantitative relation between the original and final numerosity of the three sets. No mention of the quantitative and mathematical aspects of the task was made before or during the task, and the task was not administered during math classes. Further, the Teleportation Task was the first task learners completed during this study and the learners were not told that the study had anything to do with mathematics. Given that learners were not guided towards the quan-

titative aspects of the task, it was assumed that learners who used quantitative relations in a response, spontaneously focused on the relations in that trial (also see McMullen et al., 2015).

For each of the four trials, participants' answers were scored from zero to two points, based on the extent to which they used multiplicative quantitative relations. We used the same coding scheme as McMullen and colleagues (McMullen, 2014; McMullen et al., 2015), with very good interscorer reliability (two independent raters agreed on their scoring on 98% of the items). Answers on the open-ended task were coded as relational (2 points) when they involved explicit descriptions of exact multiplicative relations (e.g., "three times more"). For the drawing task, relational responses depicted the correct number of items for all three sets of objects, based on the multiplicative relation between sets. Quasi-relational responses (1 point) involved responses that contained non-exact or incorrect descriptions of multiplicative relations (e.g., "they multiplied") on the open-ended task, and responses that depicted a consistent but incorrect multiplicative change, or a correct drawing of two out of three sets. All other responses were coded as non-relational (0 points). Finally, the sum-score of all coded responses was calculated, resulting in a minimum possible score of 0 and a maximum score of 8. This range of possible scores allowed capturing considerable variation in SFOR.

Rational Number Knowledge Test

Learners' understanding of rational numbers was measured by the Rational Number Knowledge Test (RNKT). The RNKT is a combination of two existing test instruments. The first instrument is the Rational Number Test (further abbreviated as RNT) as used in the study of McMullen et al. (2015), which included items measuring the understanding of the size and density of fractions and decimals. The reliability of this test was very high (Cronbach's alpha = .92). The second test is the Rational Number Sense Test (RNST) constructed by Van Hoof, Janssen, Verschaffel, and Van Dooren (2015). The RNST was first used in a larger research project aimed at mapping the development of rational number knowledge throughout the elementary and secondary school curriculum (Van Hoof, Verschaffel et al., 2015). It included items of the three aspects of density, size, and operations (see introduction) and every type of item was presented in its fraction and decimal form or a combination of both. The reliability of the RNST was also very high (Cronbach's alpha = .87). By complementing a subset of the RNT of McMullen et al. (2015) with a specific subset of items of the RNST of Van Hoof et al. (2015), we created an extended Rational Number Knowledge Test (further abbreviated as RNKT test). The total RNKT consisted of 24 incongruent items: 8 density items, 8 size items, and 8 operation items. Because we were interested in learners' ability to overcome the natural number bias, only incongruent items were included in the RNKT. Examples of items for all three aspects are given in Figure 1b. The reliability of the test was very high (Cronbach's alpha = .89).

Raven's Progressive Matrices

Raven's Progressive Matrices test (Raven, Court, & Raven, 1995) was used to measure

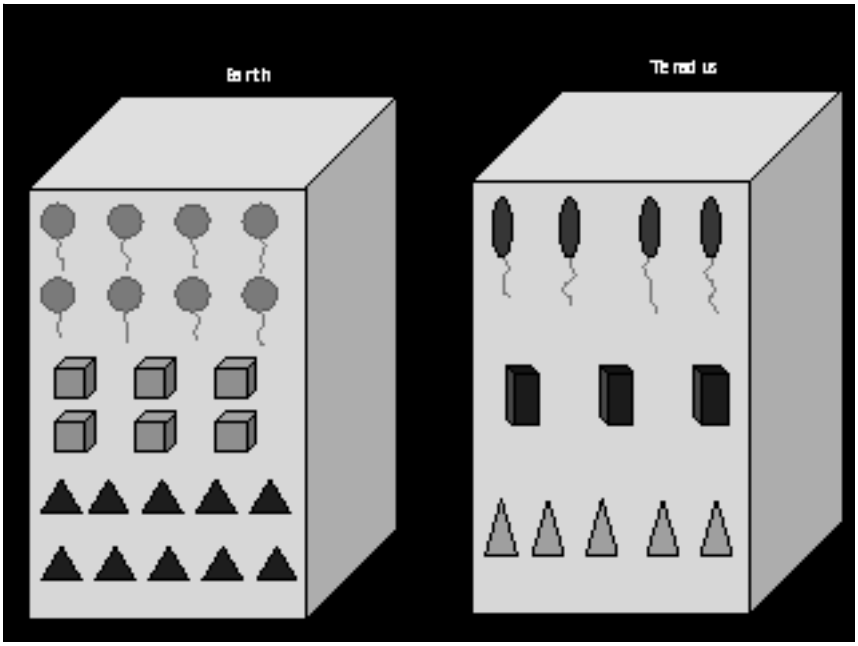

Figure 1a

Density	Size	Operations
How many numbers are there between 0.51 and 0.52?	Which is the larger number? 5/8 or 4/3	0.36 – 0.2 = …
What is the smallest possible fraction?	Which is the larger number? 0.36 or 0.5	What is half of 1/8?

Figure 1b

Figure 1 Examples of items used in the Teleportation Task to measure SFOR **a)** and in the RNKT to measure rational number knowledge **b)**

non-verbal intelligence (further abbreviated as IQ). Twenty items were used: two items from Set B (B1 and B2), the whole of Set C (C1 – C12), and six items from Set D (D1 – D6). All items were scored as correct or incorrect, leading to a maximum possible score of 20.

Arithmetical Fluency Test

Learners' arithmetical fluency was measured by means of the Tempo Test Rekenen (De Vos, 1992). This test measures the automated knowledge of the four basic opera-

tions (addition, subtraction, multiplication, and division) by means of 40 arithmetic problems on each of the four types of basic operations, as well as 40 additional arithmetic problems consisting of a mix of all the four different operations on sequential items. This leads to a total of 200 arithmetic problems. Learners were asked to complete as many arithmetic operation problems per subtest as possible in 1 minute, and as accurately as possible. Afterwards, the sum score was calculated by adding the scores on each of five subtests (i.e., one subtest per type of operation, and one subtest wherein all operations were mixed). The maximum possible score was 200.

General Mathematics Achievement Test

General mathematics achievement was measured by means of the Leerling Volg Systeem (LVS) Wiskunde (Dudal, 2003). This test contains 60 items covering several aspects of the mathematics curriculum in Flanders and is typically used by schools to monitor the progress of learners throughout primary school.

Results

Descriptive Statistics

Table 1 displays the descriptive statistics for all variables. Learners had a mean score of 4.35 on the SFOR task. There was substantial variation in learners' SFOR scores, as shown by the standard deviation (SD = 2.61) in Table 1. This was in line with previous studies of McMullen and colleagues (e.g., McMullen, 2014; McMullen et al., 2014, 2015).

The Relation Between SFOR and Rational Number Knowledge

As stated above, the main aim of this study was to further explore the relationship between learners' SFOR and their rational number knowledge. Correlation analyses confirmed that there was a strong relationship between SFOR and learners' rational number knowledge. This was the case for learners' total rational number knowledge score (r = .42, p < .001), as well as for learners'

Table 1 *Descriptive statistics*

Variables	*N*	*M*	*SD*	Range	Theoretical Maximum
SFOR	355	4.35	2.61	0-8	8
IQ	349	14.94	3.28	3-20	20
Arithmetical fluency	349	111.98	23.23	40-191	200
Rational number knowledge	353	9.47	4.98	0-23	24
Density	353	0.93	1.25	0-7	8
Size	353	5.14	2.58	0-8	8
Operations	353	3.40	2.15	0-8	8
Math achievement	345	42.67	8.77	5-59	60
Age in months	356	135.08	11.34	117-164	/

scores on the separate aspects: learners' score on the size tasks ($r = .34$, $p < .001$) and learners' score on the operation tasks ($r = .41$, $p < .001$). A smaller, yet significant correlation was found between SFOR and learners' score on the density tasks ($r = .26$, $p < .001$).

Predicting Learners' Rational Number Knowledge

We conducted several linear regression analyses to further investigate the relation between learners' SFOR and their rational number knowledge. In the first regression analysis, in an attempt to confirm McMullen and colleagues' (2015) previous findings, learners' arithmetical fluency was entered as a first predictor of learners' rational number knowledge, followed by learners' IQ, and finally SFOR. In three additional regression analyses, every aspect of rational number knowledge (density, size, and operations) was entered separately as dependent variable to define the predictors of these three sub-aspects separately. The results of the four linear regression analyses are given in Table 2. This table also contains the R^2 change and standardized beta for every predictor, in order to show the amount of explained variance added by each predictor, and the unique contribution of each predictor.

Results of the first regression analysis show that, after including arithmetical fluency and IQ in the model, learners' SFOR still accounted for 8% additional variance in their total rational number knowledge score ($p < .001$). The total amount of variance of learners' rational number knowledge explained by the three predictors was 34%. Next to arithmetical fluency, SFOR explained the largest amount of variance. The same trends were found for density: After including arithmetical fluency and IQ in the model, SFOR still accounted for 3% variance in learners' density score ($p < .01$). The total amount of variance in density explained by the three predictors was 14%. Also for size, results of the second additional regression analysis indicated that after including arithmetical fluency and IQ in the model, SFOR still accounted for 5% variance in size ($p < .001$). The total amount of variance of learners' size score explained by the three predictors was 24%. Finally, for the operations aspect, results indicated that after including arithmetical fluency and IQ in the model, SFOR still

Table 2 *Predictors of rational number knowledge*

Variable	Rational number knowledge		DENSITY		SIZE		OPERATIONS	
	Standardized *β*	R^2 Change	Standardized *β*	R^2 Change	Standardized *β*	R^2 Change	Standardized *β*	R^2 Change
Arithmetical fluency	.41***	.24***	.27***	.10***	.37***	.18***	.36***	.19***
IQ	.04	.02*	.05	.01	.03	.01*	.03	.01*
SFOR	.30***	.08***	.18**	.03**	.24***	.05***	.31***	.09***
Total R^2		.34***		.14***		.24***		.29***

Note. * p < .05, ** p < .01, *** p < .001

accounted for 9% variance in learners' operations score ($p < .001$). The total amount of variance of operations explained by the three predictors was 29%.

As stated above, we further investigated whether the relation between learners' SFOR and their rational number knowledge would still hold after controlling for learners' general mathematics achievement and age, next to the aforementioned control variables already used by McMullen and colleagues (e.g., McMullen, 2014). Results showed that, after including learners' age in months, arithmetical fluency, general math achievement, and IQ in the model, SFOR still accounted for 1% variance in learners' rational number knowledge scores (see Table 3). The total amount of variance of learners' rational number knowledge explained by the five predictors was 55%. These results clearly showed that after including both learners' age and their general math achievement, the role of SFOR as predictor of learners' rational number knowledge considerably decreased. However, it is important to note that, after controlling for learners' age in months, arithmetical fluency, general math achievement, and IQ, SFOR still significantly predicted RNKT ($\beta = .12, p < .05$). Further, if we examined the three different aspects separately, we saw (see Table 3) that – after controlling for learners' age in months, arithmetical fluency, general math achievement, and IQ – SFOR only significantly predicted the aspect of operations ($\beta = .14, p < .01$), while this was not the case for the aspects of density and size (both p-values > .05).

Discussion

In the present study we elaborated on previous research by further exploring the relationship between learners' SFOR and their rational number knowledge. This study not only investigated whether SFOR related to rational number knowledge as a whole, but also to the several aspects of the natural number bias. We additionally extended previous research by 1) including test items of operations with rational numbers in the Rational Number Knowledge Test, which was an aspect of rational number knowledge that

Table 3 *Predictors of rational number knowledge*

Variable	Rational number knowledge		DENSITY		SIZE		OPERATIONS	
	Standardized β	R^2 Change	Standardized β	R^2 Change	Standardized β	R^2 Change	Standardized β	R^2 Change
Months	.51***	.32***	.35***	.13***	.39***	.19***	.50***	.31***
Arithmetical fluency	.20***	.12***	.12*	.05***	.19***	.10***	.16***	.08***
General math achievement	.30***	.10***	.25***	.06***	.25***	.07***	.23***	.07**
IQ	.03	.00	.01	.00	.03	.00	.03	.00
SFOR	.12*	.01*	.04	.00	.09	.01	.14**	.02**
Total R^2		.55***		.24***		.36***		.48***

Note. * p < .05, ** p < .01, *** p < .001

was missing in the instrument used by McMullen and colleagues (2015) and 2) controlling for learners' general mathematics achievement and age, which was not done by McMullen and colleagues (2015).

First, based on the data of 356 upper elementary school children, we found that SFOR indeed correlated highly with learners' rational number knowledge (as measured by incongruent rational number tasks, which are those rational number tasks in which natural number knowledge may interfere in obtaining the correct answer).

Second, several regression analyses revealed that learners' SFOR relates to their rational number knowledge, as well as to each of the three aspects of the natural number bias, after controlling for arithmetical fluency and non-verbal intelligence. While this relation was already shown by McMullen and colleagues (e.g., McMullen, 2014; McMullen et al., 2015) for rational number knowledge as a whole, the present study successfully replicated this finding, and moreover showed that SFOR also relates to the three aspects of the natural number bias separately. The predictive value of SFOR was the smallest on the density aspect (R^2 change = 3%; β = .18) followed by the size aspect (R^2 change = 5%; β = .24) and the largest for the operations aspect (R^2 change = 9%; β = .31).

Third, although including both learners' age and their general math achievement led to a decrease of the unique contribution of SFOR as a predictor of learners' rational number knowledge, this relation remained significant. Further, a separate analysis for the three different aspects of the natural number bias revealed that after controlling for learners' age in months, arithmetical fluency, general math achievement, and IQ, SFOR only significantly predicted the understanding of the aspect of operations, while this result could not be found in the aspects of density and size.

In what follows, we will further discuss those results and their impact on future research. First, we will discuss the relation between SFOR and learners' rational number knowledge. Second, we will elaborate on our results concerning the relation between SFOR and each of the three aspects of the natural number bias.

First, our results concerning the relation between SFOR and rational number knowledge imply that learners who spontaneously pay more attention to quantitative relations in non-mathematical settings also have a better understanding of rational numbers. In this case, paying special attention to multiplicative quantitative relations is relevant, as rational numbers are in fact a multiplicative quantitative relation (a ratio) between two whole numbers.

The relation between SFOR and rational number knowledge may further be explained by the idea that, due to more frequent self-initiated practice, learners with a high SFOR tendency have more everyday experiences with (multiplicative) quantitative relations (McMullen, 2014). These experiences may call for reasoning about non-natural numbers and approximate quantitative relations. McMullen and colleagues (2015) give the example of a child who realizes that sharing two cookies among three friends results in a non-natural number answer and requires approximate division into thirds. As a result, learners with a high SFOR tendency may more frequently experience that the natural number system has its limitations and that other kinds of numbers are required to model some situations. These learners might be less inclined to form the

above-mentioned intuitive idea of a number as a natural number as it was described in the introduction, or they might have more experiences that question this intuition once it is developed. This can be fruitful in understanding the rational number system, especially in overcoming the natural number bias (McMullen et al., 2015).

The aforementioned explanation for our finding that learners with high SFOR have a better understanding of rational numbers (and more specifically in those tasks in which natural number reasoning leads to an incorrect answer) requires further research. This research may explore whether the observed correlation between SFOR and rational number knowledge is a causal one, by means of (quasi-) experimental studies that investigate if a training program that stimulates focusing on quantitative relations would facilitate the development of rational number knowledge.

Our findings are not only important for further research on the role of the natural number bias in the development of rational number knowledge; they also demonstrate the relevance of the SFOR tendency itself. They show that the spontaneous inclination to focus on quantitative relations in non-mathematical settings is an important source of individual differences in a more advanced mathematical domain, namely that of rational numbers. However, in line with previous work on learners' SFOR tendency (McMullen, 2014; McMullen et al., 2013, 2014), the present study exclusively focused on quantitative relations that are multiplicative in nature. The question remains to what extent learners also spontaneously would focus on other types of quantitative relations, such as additive relations (see Degrande, Verschaffel, & Van Dooren, 2015; McMullen, 2014), and to what extent these distinct foci on different types of quantitative relations would be differentially related to their rational number knowledge and other mathematical concepts and skills.

Second, SFOR not only significantly predicted learners' rational number knowledge as a whole; it also significantly predicted all three aspects of the natural number bias, after controlling for arithmetical fluency and non-verbal intelligence. However, after additionally taking into account learners' age and general math achievement, SFOR only significantly predicted learners' knowledge of operations – and not the two other aspects of the natural number bias. These findings might suggest that the three aspects of the natural number bias are not to the same extent impacted by learners' age and math achievement.

These results further reveal some information about the instrument used to measure learners' SFOR. In particular, the finding that SFOR only significantly predicted the understanding of the aspect of operations may be explained by the SFOR-instrument. Learners were given a high SFOR score when they correctly noticed that the objects were multiplied or divided in the teleportation machine, and either expressed this specific operation explicitly (in the open-ended trials) or replicated that operation in another trial (in the drawing trials). In order to obtain the maximum SFOR-score (see coding scheme above), one thus had to think about the multiplicative relations underlying the transformations, and mathematically express those relations by means of operations. Therefore, it is not that surprising that SFOR only predicted the understanding of the aspect of operations (after controlling for arithmetical fluency, non-verbal intelligence, age, and

general math achievement) and not the aspects of size and density. The latter two aspects rather measured the conceptual knowledge of rational numbers, while operations mainly measured the procedural knowledge of rational numbers.

In sum, the present study further explored the relation between learners' SFOR and their rational number knowledge, more specifically the three aspects of the natural number bias (i.e., density, size and operations). Our results do not only contribute to our understanding of the process of acquiring rational number knowledge, they also demonstrate the relevance of SFOR. Nevertheless, some important theoretical issues remain to be clarified in future research. Providing an answer to those issues would not only help get a better view on the impact of SFOR – in its different conceptualizations and operationalizations – on the understanding of various mathematical domains, but also to develop targeted pedagogical interventions for learners in different grades.

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